Fine-Grained Complexity Analysis of Dependency Quantified Boolean Formulas (Extended Abstract)

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Abstract

Dependency Quantified Boolean Formulas (DQBF) extend Quantified Boolean Formulas by allowing each existential variable to depend on an explicitly specified subset of the universal variables. The satisfiability problem for DQBF is NEXP-complete in general, with only a few tractable fragments known to date. We investigate the complexity of DQBF with k existential variables (k-DQBF) under structural restrictions on the matrix—specifically, when it is in Conjunctive Normal Form (CNF) or Disjunctive Normal Form (DNF)—as well as under constraints on the dependency sets. For DNF matrices, we obtain a clear classification: 2-DQBF is PSPACE-complete, while 3-DQBF is NEXP-hard, even with disjoint dependencies. For CNF matrices, the picture is more nuanced: we show that the complexity of k-DQBF ranges from NL-completeness for 2-DQBF with disjoint dependencies to NEXP-completeness for 6-DQBF with arbitrary dependencies.

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Related Version *Full Version*: [3]

1 Introduction

In this paper, we studied the complexity of Dependency Quantified Boolean Formulas (DQBF) under different structural constraints. In particular, we define the subclasses k-DQBF$_\beta^\alpha$ of DQBF, where $k \geqslant 1$ indicates the number of existential variables, $\alpha \in \{\mathsf{d}, \mathsf{de}, \mathsf{dec}, \mathsf{ds}\}$ indicates the condition on the dependency sets, and $\beta \in \{\mathsf{cnf}, \mathsf{dnf}\}$ indicates that the matrix is in conjunctive/disjunctive normal form. Let $\bar{x}$ be the set of universal variables, and $D_i \subseteq \bar{x}$ be the dependency set of the i-th existential variable. The annotation α is defined by

DQBF$^{\mathsf{d}}$ For every $i \neq j$, $D_i \cap D_j = \emptyset$,
DQBF$^{\mathsf{de}}$ For every $i \neq j$, $D_i \cap D_j = \emptyset$ or $D_i = D_j$,
DQBF$^{\mathsf{dec}}$ For every $i \neq j$ with $|D_i| \leqslant |D_j|$, $D_i \cap D_j = \emptyset$, $D_i = D_j$, or $D_j = \bar{x}$, and
DQBF$^{\mathsf{ds}}$ For every $i \neq j$ with $|D_i| \leqslant |D_j|$, $D_i \cap D_j = \emptyset$ or $D_i \subseteq D_j$.

It is previously proven that DQBF$_{\mathsf{cnf}}^{\mathsf{de}}$ is Σ_3^{P}-complete [6], and that k-DQBF is coNP-, PSPACE-, and NEXP-complete for $k = 1$, $k = 2$, and $k \geqslant 3$, respectively [5].

Our results, summarised in Table 1, help map out the complexity of natural fragments of DQBF and show how both the formula structure and dependency restrictions play a key role in determining tractability.

Table 1 Summary of the complexity results.

<table>
<tr><th>k</th><th>k-DQBF$^{\mathsf{d}}_{\mathsf{cnf}}$</th><th>$k$-DQBF$^{\mathsf{de}}_{\mathsf{cnf}}$</th><th>$k$-DQBF$^{\mathsf{dec}}_{\mathsf{cnf}}$,
k-DQBF$^{\mathsf{ds}}_{\mathsf{cnf}}$</th><th>$k$-DQBF$_{\mathsf{cnf}}$</th><th>$k$-DQBF$^{\mathsf{d}}_{\mathsf{dnf}}$</th></tr>
<tr><td>1</td><td>-</td><td>-</td><td>-</td><td>L</td><td>coNP-c</td></tr>
<tr><td>2</td><td>NL-c</td><td>NL-c</td><td>NL-c</td><td>coNP-c</td><td>PSPACE-c</td></tr>
<tr><td>$3, 4$</td><td rowspan="4">NP-c</td><td rowspan="3">NP-c</td><td rowspan="3">NP-c</td><td>Π_2^{P}-h</td><td rowspan="4">NEXP-c</td></tr>
<tr><td>5</td><td>PSPACE-h</td></tr>
<tr><td>6+</td><td rowspan="2">NEXP-c</td></tr>
<tr><td>Non-const.</td><td>Σ_3^{P}-c [6]</td><td>NEXP-c [6]</td></tr>
</table>

Note: "-c" denotes "-complete" and "-h" denotes "-hard."

2 Summary of Results and Proof Techniques

For $\mathsf{sat}(k\text{-DQBF}^{\alpha}_{\mathsf{dnf}})$, even the most restricted dependency structure d matches the complexity of general k-DQBF. To prove this, we need the following observations. First, the 2- and 3-DQBF formulas constructed in the hardness proof in [5] already have the dependency structure d. In addition, a DNF version of Tseitin transformation [2, Proposition 1] allows us to transform any k-DQBF$^{\alpha}$ into an equisatisfiable k-DQBF$^{\alpha}_{\mathsf{dnf}}$ without changing any of the dependency sets.

The complexity of $\mathsf{sat}(k\text{-DQBF}^{\mathsf{d}}_{\mathsf{cnf}})$ (resp., $\mathsf{sat}(\text{DQBF}^{\mathsf{d}}_{\mathsf{cnf}})$) matches that of k-SAT (resp., SAT), i.e., when restricted to CNF matrices and pairwise disjoint dependency structures, the problem becomes exponentially easier. The membership proof makes use of the universal expansion of a k-DQBF$^{\mathsf{d}}_{\mathsf{cnf}}$ formula Φ into the (exponentially large) SAT formula $\exp(\Phi)$. The set of falsifying assignments to the CNF matrix can grouped by the clause they falsify, so the instantiated clauses in $\exp(\Phi)$ can be represented as the union of polynomially many sets. Moreover, the disjoint dependency structure allows us to further represent each of these sets succinctly. We can therefore check the satisfiability of Φ on the succinctly represented version of $\exp(\Phi)$. The hardness proof is simply by a reduction from k-SAT, where the variables are encoded into a logarithmic number of universal variables, and the k existential variables are k copies of the same assignment that are used to ensure each clause is satisfied by at least one literal in the clause.

The result is further generalized to $\mathsf{sat}(k\text{-DQBF}^{\alpha}_{\mathsf{cnf}})$ for constant k and $\alpha \in \{\mathsf{de}, \mathsf{dec}, \mathsf{ds}\}$. The laminar structure of ds allows us to iteratively perform universal reduction [1, 4], variable elimination by resolution [7], and splitting the formula into disjoint subformulas, until the resulting formula is a SAT formula. This gives a stark contrast to the results shown in [6], where $\mathsf{sat}(\text{DQBF}^{\mathsf{de}}_{\mathsf{cnf}})$ is Σ_3^{P}-complete and $\mathsf{sat}(\text{DQBF}^{\mathsf{dec}}_{\mathsf{cnf}})$ is NEXP-complete.

The complexity of $\mathsf{sat}(k\text{-DQBF}_{\mathsf{cnf}})$ is the most challenging and remains partly unanswered. We show that $\mathsf{sat}(2\text{-DQBF}_{\mathsf{cnf}})$ is coNP-complete. The membership is given by an NP algorithm for its falsity which guesses the assignment to the common dependency first and solves the induced 2-DQBF$^{\mathsf{d}}_{\mathsf{cnf}}$ formula in polynomial time. The hardness is given a reduction from 3-DNF tautology. A similar technique is used to show the Π_2^{P}-hardness of $\mathsf{sat}(3\text{-DQBF}_{\mathsf{cnf}})$ by a reduction from Π_2^{P}-QBF. Finally, we reformulated Tseitin transformation to use three additional existential variables instead of linearly many. This allows us to lift the complexity of $\mathsf{sat}(k\text{-DQBF})$ to $\mathsf{sat}((k+3)\text{-DQBF}_{\mathsf{cnf}})$, giving us the PSPACE-hardness of $\mathsf{sat}(5\text{-DQBF}_{\mathsf{cnf}})$ and the NEXP-completeness of $\mathsf{sat}(6\text{-DQBF}_{\mathsf{cnf}})$.

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